Math 241

Problem Set 9 solution manual

Exercise. A9.1

a- Dodecahedron: vertices =20, edges=30 , faces=12 , and 3 edges emanate from each vertex. Icosahedron: vertices =12, edges=30, faces=20 , and 5 edges emanate from each vertex.

b- For the dodecahedron:
Notice that the stabilizer of a vertex is a cyclic group of order 3, and hence we can deduce that the order of the group G_d is equal to 3×number of vertices, i.e $|G_d|$=60. If we allow reflections then the stabilizer of the vertex of a dodecahedron is isomorphic to S_3, and hence the number of elements would be 120.

For the icosahedron:
Similarly as above, one can notice that the stabilizer of a vertex is the cyclic group of order 5, hence $|G_i|$=60, and allowing reflections then the stabilizer of a vertex would be isomorphic to D_5, hence the number if elements would be 120.

c- For the icosahedron:(the stab of a vertex is computed in part (b))
For the face we have the stabilizer is a cyclic group of order 3.
For the edge we have the stabilizer is a cyclic group of order 2.

For the dodecahedron :(the stab of a vertex is computed in part (b))
For the face we have the stabilizer is a cyclic group of order 5.
For the edge we have the stabilizer is a cyclic group of order 2.

Now it is easy to check that those results are compatible with the number of elements in G_i , and G_d computed in part (b).

d- To see that the two groups G_i, and G_d are isomorphic, we have to consider each vertex of the icosahedron as a face of dodecahedron, and each face of the icosahedron as a vertex of the dodecahedron.

As for seeing the isomorphism between G_d and A_5, you can either use the action of G_i on the 5 squares that can be embedded inside the dodecahedron, or you can find a a subgroup H of order 12 in G_d, and use the action of G_d on G_d/H.

Exercise. A9.2

For S_5:
Since we know that $f(ijk..)f^{-1} = (f(i)f(j)f(k)...)$, then it is easy to see that all the m-cycles in S_5 are conjugate. Similarly the $f(ij...)(kl...)f^{-1} = (f(i)f(j)...)(f(k)f(l)...)$, and hence all products of m-cycles by n-cycles in S_5 are conjugate. Hence we have the following:

x	C_x	$\|C_x\|$	$\|Z_x\|$
id_{S_5}	$\{id\}$	1	120
(ij), transposition	{ all transpositions }	10	12
(ijk) 3-cycle	{ all 3-cycles }	20	6
$(ij)(kl)$ product of two disjoint transpositions	{ all products of two disjoint transpositions }	15	8
$(ijkl)$ 4-cycle	{all 4-cycles}	30	4
$(ijk)(lm)$ product of disjoint transposition and 3-cycle	{ all disjoint products of transpositions with 3-cycles }	20	6
$(ijklm)$, 5-cycle	{ all 5-cycles }	24	5

Lemma 1. If τ commutes with σ, then $(f\tau)\sigma(f\tau)^{-1} = f\sigma f^{-1}$.

Proof. $(f\tau)\sigma(f\tau)^{-1} = (f\tau)\sigma\tau^{-1}f^{-1} = f(\tau\sigma\tau)f^{-1} = f\tau\tau\sigma f^{-1} = f\sigma f^{-1}$.

Corollary 1. If $\exists\ \tau$ (transposition) that commutes with σ then the conjugacy class of σ in A_5 is the same as the conjugacy class in S_5.

Proof. It is easy to see that any element conjugate to σ in A_5 is conjugate to σ is S_5. Now suppose σ' is conjugate to σ in S_5, then $\exists\ f \in S_5$ such that $f\sigma f^{-1} = \sigma$, if $f \in A_5$ then σ and σ' are conjugate in A_5, if not (i.e $f \notin A_5$) then $f\tau\sigma_1\tau f^{-1} = \sigma_2$ and $f\tau$ is an element of A_5, hence $\sigma$is conjugate to σ' is A_5.

So for A_5 we have only 5 orbits. In A_5 we have 3-cycles, product of disjoint transpositions, and 5-cycles.

For any 3-cycle we can find a transposition τ that commutes with it, and hence by above corollary the conjugacy class for the 3-cycles is the same as in S_5.

For the product of two disjoint transpositions, Notice that (ij) commutes with $(ij)(kl)$ for distinct i, j, k, l, and hence their conjugacy class is the same as in S_5.

For the 5-cycles they are not all conjugate. Let C_σ be the conjugacy class in S_5, and C'_σ be the conjugacy class in A_5. Fix a transposition τ , it is easy to see that $S_5 = A_5 \cup A_5\tau$, and we can then deduce that $C_\sigma = C'_\sigma \cup C'_{\tau\sigma\tau^{-1}}$ since for $f \in S_5$ with $f\sigma f^{-1}$ either $f \in A_5$, and hence $f\sigma f^{-1} \in C'_\sigma$ or $f \notin A_5$, then $\exists\ g \in A_5$ such that $f = g\tau$, with $f\sigma f^{-1} = (g\tau)\sigma(g\tau)^{-1}$, so $f\sigma f^{-1} \in C'_{\tau\sigma\tau^{-1}}$.

Next we find $|C'_\sigma| = |A_5|/|Z'_\sigma|$, where $Z'_\sigma = Z_\sigma \cap A_5$. But $Z'_\sigma =< \sigma >$ since from above table we have $|Z_\sigma| = 5$, and we know that $< \sigma >\subset Z'_\sigma$. Hence we deduce that $|C'_\sigma| = 12$, and similarly $|C'_{\tau\sigma\tau^{-1}}|$=12.

Hence we deduce that we have two conjugacy classes for the 5-cycles.

Exercise. A9.3

Consider the function : $f : GL_n(\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p^*$ defined by $f(g) = |g|$.

It is easy to see that f is a surjective group homomorphism.

The kernel of f is equal to the subgroup $SL_n(\mathbb{Z}_p)$.

So we deduce that $GL_n(\mathbb{Z}_p)/SL_n(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$, hence $|SL_n(\mathbb{Z}_p)| = |GL_n(\mathbb{Z}_P)|/|\mathbb{Z}_p^*| = (p^n - 1)(p^n - p)...(p^n - p^{n-1})/(p - 1)$.

Exercise. A9.4

Lemma 2. If L_1, L_2, and L_3 are three different lines in the plane then $\exists\, v_1 \in L_1$, and $v_2 \in L_2$ such that $v_3 = v_1 + v_2 \in L_3$ and $L_1 = span\{v_1\}$, $L_2 = span\{v_2\}$, and $L_3 = span\{v_3\}$.

Proof. Suppose that $L_1 = span\{u_1\}$, $L_2 = span\{u_2\}$, and $L_3 = span\{u_3\}$, then u_1, and u_2 are linearly independent, and hence $u_3 = iu_1 + ju_2$ (i, j both non-zero). Then we let $v_1 = iu_1$, and $v_2 = ju_3$, and we get $v_3 = v_1 + v_2$.

A one dimensional subspaces of $\mathbb{R}^2$ is the span of some vector (a, b). Let L_1, L_2, and L_3, be 3 different lines in $\mathbb{R}^2$, hence we can find 3 vector v_1, v_2, and v_3 such that $span\{v_i\} = L_i$, with $v_3 = v_1 + v_2$. Similarly for the three lines m_1, m_2, and m_3, we can find 3 vectors w_1, w_2, w_3 such that $w_3 = w_1 + w_2$, and $m_i = span\{w_i\}$.

Next we can find a matrix $g \in GL(2, \mathbb{R})$ such that $gv_1 = w_1$, and $gv_2 = w_2$, where g is the transition matrix from the basis $\{v_1, v_2\}$ to $\{w_1, w_2\}$.

So we have $gv_3 = g(v_1 + v_2) = w_1 + w_2 = w_3$. So we get our result.

Exercise. A9.5

Let us consider first the $L = span\{u\}$, where $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (Note that $L = span\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \}$ for any $a \neq 0$).

The orbit of L is $O_L = \{g.L \mid g \in B\}$. Let $g = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with a, c non-zero, then $g.L = span(gu) = span(\begin{bmatrix} a \\ 0 \end{bmatrix}) = span(u) = L$, so $O_L = \{L\}$.

This orbit has one element whose stabilizer is B.

For any other line L such that $L = span(v)$, where $v = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ with $\alpha, \beta \in \mathbb{R}$, and $\beta \neq 0$, we have $g.L = span(gv) = span(\begin{bmatrix} a\alpha + b\beta \\ c\beta \end{bmatrix}$. Notice that for any vector $\begin{bmatrix} i \\ j \end{bmatrix} \in \mathbb{R}^2$ with $j \neq 0$, we can choose $g = \begin{bmatrix} 1 & \frac{i-\alpha}{\beta} \\ 0 & \frac{j}{\beta} \end{bmatrix}$, then $g.L = span(\begin{bmatrix} i \\ j \end{bmatrix})$. We deduce that O_L is equal to all one dimensional subspaces of $\mathbb{R}^2$ except the one spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The stabilizer of $L = span(v)$ where $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the set of diagonal matrices.

Now generalizing for the action of the group of upper triangular matrices B of $GL(n, \mathbb{R})$.

We start by $u = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $L = span(u)$. The orbit of L is $O_L = \{L\}$.

Next the we consider the vector $v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_2 \neq 0$, and we let

$L = span(v)$, the orbit of L is equal to all the one dimensional subspace generated by vectors $w = \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_2 \neq 0$.

We keep doing this and hence we get n different orbits, where the ith orbit is the set of the lines spanned by a vector of the form $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, with $\alpha_i \neq 0$.

Section. 36

Exercise. 1
$p = 3$, and $|G| = 12 = 2^2 \times 3$, hence the order of the Sylow 3-subgroup is 3.

Exercise. 2
$p = 3$, and $|G| = 54 = 2 \times 3^3$, hence the order of the Sylow 3-subgroup is 27.

Exercise. 3
$p = 2$, and $|G| = 24 = 2^4 \times 3$, let s be the number of the Sylow 2-subgroups, we know from the third Sylow theorem that $s/24$,and $s \equiv 1$ mod(2), the divisors of 24 are 1,2,3,4,6,8,12,24, but s must be congruent to 1 mod(2), so our only choices are 1, and 3.

Exercise. 4
$|G| = 255 = 3 \times 5 \times 17$, following the same argument as in ex 3, the number of Sylow 3-subgroups can be either 1 or 85, and the number if the Sylow 5-subgroups can either be 1 or 51.

Exercise. 11

$G_H\{g \in G \mid gHg^{-1} = H\}$.

a- identity is in G_H since $eHe^{-1} = eHe = H$.

b- G_H is closed under multiplication. Let $g, l \in G_H$, then $gHg^{-1} = H$, and $lHl^{-1} = H$, and hence $(gl)H(gl)^{-1} = g(lHl^{-1})g^{-1}) = gHg^{-1} = H$. So $gl \in G_H$.

c- Let $a \in G_H$ then $aHa^{-1} = H$, but also $H = (a^{-1}a)H(a^{-1}a) = a^{-1}(aHa^{-1})a = a^{-1}Ha$, hence $a^{-1} \in G_h$.

Hence G_H is a subgroup of G.

Exercise. 12

G has a unique Sylow p-subgroup called P. Let $g \in G$ be any element gPg^{-1} is another Sylow p-subgroup , hence $gPg^{-1} = P$, since P is unique, and hence all the conjugates of P are equal to P, so P is normal. Then G has a non-trivial normal subgroup, so G is not simple.

Exercise. 13

$|G| = 45 = 3^2 \times 5$.

G has a Sylow 3-subgroup of order 9, using the same argument used in number 3 we can deduce that it is a unique subgroup, and hence by number 12 it is normal. So G has a normal subgroup of order 9.